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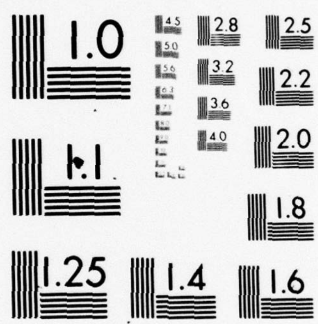

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A Double Integral Quadratic Cost Problem with Application to Feedback Stabilization *

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Abstract

This paper considers an optimal regulator problem which is different from the conventional linear quadratic cost problem but leads to a stable linear feedback control. The problem considers a double integral quadratic cost function and an integral constraint on state trajectories. The optimal open-loop control is transformed to a closed-loop control and subsequently a modified control is obtained based on a receding horizon notion. This modified control law is shown to be asymptotically stable and to result in a new method for stabilizing linear time-varying systems, in addition to the methods of [1] and [2], as well as providing an easy means to stabilize time-invarient systems comparable to the method of [3]. Moreover, the gain matrix for the modified control is obtained from a Riccati-type equation over a finite time interval, and a large class of nonlinearities can be allowed in the closed-loop without destroying its stability.

I. Introduction

Consider a linear time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1.1}$$

$$y(t) = C(t)x(t) \tag{1.2}$$

where $x(t_o) = x_o$, and $(A(t), B(t), C(t))$ are nxn, nxm and pxn piecewise continous matrices. There have been many studies in obtaining stable linear feedback controls with some optimal basis. The most familiar of these is obtained from the steady state optimal control for the quadratic cost

$$J(u) = \int_{t_o}^{t_f} [y'(t)Q(t)y(t) + u'(t)R(t)u(t)]dt \tag{1.3}$$

with a free terminal condition, where $Q(t)$ and $R(t)$ are piecewise continuous weighting matrices with $Q(t) \gtrsim 0$ and $R(t) > 0$ [1]. Recently in [2] we suggested another stable linear feedback control which is based on a receding horizon notion applied to the above quadratic cost problem with a fixed terminal constraint

$$x(t_f) = 0. \tag{1.4}$$

This is equivalent to the following integral constraint on the control

$$\int_{t_o}^{t_f} \Phi(t_o,t)B(t)u(t)dt = - x(t_o), \tag{1.5}$$

where $\Phi(t,\tau)$ is the state transition matrix for (1.1). We have shown in [2] that this control law, which is obtained from the above problem by a receding horizon notion ($t_f = t + T$), stabilizes the linear system (1.1) under certain conditions, and that the gain matrix is obtained from a Riccati differential equation defined over a finite time interval of arbitrarily short duration. Such a receding horizon notion has also been discussed by Thomas for time invariant systems and with $Q = 0$ [10].

In addition to stability, the well-known free terminal steady state optimal control of [1,4] possesses the important property that it tolerates certain nonlinearities in the feedback loop without destroying stability. Specifically, the resulting system has been shown in [5,9] to possess infinite gain margin and a 60° phase margin under certain conditions which include $Q(t) \geq \alpha I$ for some $\alpha > 0$. The modified receding horizon control in [2] can be shown to possess the same property. These kinds of relationships between system coefficients and the tolerance of nonlinearities in the feedback loop have been extensively studied, particularly the absolute stability problem for time invariant systems [7,8], commencing with Luré.

The purpose of this paper is to introduce an optimal regulator problem which is different from the conventional linear quadratic cost problem but whose modification leads to another stable linear feedback control with tolerance for a large class of nonlinearities in the feedback loop. In Section II a double integral control energy problem with a compatible constraint is formulated and its optimal open-loop control solution is transformed to a closed-loop linear state feedback control. In Section III it is shown that the modified control law obtained by the receding horizon notion is a uniformly asymptotically stable control which allows for a large class of nonlinearities in the feedback loop. The corresponding results for linear time invariant systems are stated in Section IV.

II. A Double Integral Minimum Control Energy Problem

For the given system (1.1)-(1.2), we consider a double integral control energy function

$$J(u) = \int_{t_o}^{t_f} \int_{t_o}^{\tau} u'(s)R(s)u(s)\,ds\,d\tau \tag{2.1}$$

and a constraint which is similar to (1.5) but is compatible with the cost function (2.1):

$$\int_{t_o}^{t_f}\int_{t_o}^{\tau} \Phi(t_o,s)B(s)u(s)ds\,d\tau = -x(t_o). \tag{2.2}$$

From the variation of constants formula, the constraint (2.2) on the control is equivalent to the following integral constraint on the state:

$$x(t_o) + \int_{t_o}^{t_f} \Phi(t_o,s)x(s)ds = (t_f - t_o)x(t_o). \tag{2.3}$$

The solution to the above problem is given as follows.

Theorem 2.1. The optimal control law for the system (1.1) which minimizes the cost function (2.1) subject to the constraint (2.2) (or equivalently (2.3)) is given by

$$u(t) = -R^{-1}(t)B'(t)\Phi'(t_o,t)\tilde{P}^{-1}(t_o,t_f)x(t_o) \tag{2.4}$$

where

$$\tilde{P}(t_o,t_f) = \int_{t_o}^{t_f}\int_{t_o}^{\tau} \Phi(t_o,s)B(s)R^{-1}(s)B'(s)\Phi'(t_o,s)ds\,d\tau. \tag{2.5}$$

Proof: The proof is straightforward by introducing a Lagrange multiplier λ for the constraint (2.2). The necessary condition for minimizing the cost function

$$J_\lambda(u) = \int_{t_o}^{t_f}\int_{t_o}^{\tau} \{\tfrac{1}{2}u'(s)R(s)u(s) + \lambda'\Phi(t_o,s)B(s)u(s)\}ds\,d\tau$$

is that $\frac{\partial J_\lambda}{\partial u} = 0$, i.e.,

$$R(s)u(s) + B'(s)\Phi'(t_o,s)\lambda = 0.$$

Thus the optimal control is given by $u(s) = - R^{-1}(s)B'(s)\Phi'(t_o,s)\lambda$. Combining this control and the constraint (2.2) yields $\lambda = \tilde{P}^{-1}(t_o,t_f)x(t_o)$, which completes the proof.

The double integral matrix $\tilde{P}(t_o,t_f)$ can be expressed as

$$\tilde{P}(t_o,t_f) = \int_{t_o}^{t_f} P(t_o,\tau)d\tau \qquad (2.6)$$

where the controllability matrix $P(t_o,\tau)$ is obtained from

$$- \frac{\partial}{\partial s} P(s,\tau) = -A(s)P(s,\tau) - P(s,\tau)A'(s) + B(s)R^{-1}(s)B'(s) \qquad (2.7)$$

with the boundary condition $P(\tau,\tau) = 0$. By combining (2.1) and (2.4), the optimal cost is given by

$$J(u) = x'(t_o)\tilde{P}^{-1}(t_o,t_f)x(t_o).$$

Since the controls in (2.1) are more heavily weighted around the initial time than the terminal time, it is believed that the control mahnitude of (2.4) tends to be smaller than that of a conventional optimal control solution near the initial time. It is noted that the optimal open-loop control (2.4) does not guarantee the terminal constraint $x(t_f) = 0$ because the constraint (2.3) does not necessarily imply (1.4).

A closed-loop control can be defined from the open-loop control (2.4) by replacing t_o with t as follows:

$$u(t) = -R^{-1}(t)B'(t)\tilde{P}^{-1}(t,t_f)x(t) \qquad (2.8)$$

where $\tilde{P}(t,t_f)$ is given by either (2.5) or (2.6) with t_o replaced by t. It is noted that the closed-loop control (2.8) tries to satisfy the sliding constraint

$$x(t) + \int_t^{t_f} \Phi(t,s)x(s)ds = (t_f - t)x(t) \qquad (2.9)$$

at each instant "t". From this constraint it is readily seen that the closed-loop control (2.8) guarantees the terminal constraint $x(t_f) = 0$, in contrast with the open-loop control (2.4), as can be seen by letting t approach t_f in (2.9). It is the control (2.8) that will be modified by the receding horizon notion in order to obtain a stable linear feedback control which satisfies the properties mentioned in the Abstract.

III. A Modified Regulator for Time-Varying Systems

In addition to the infinite-time optimal control of [1,4] and the modified receding horizon control of [2], we introduce in this section another stable linear feedback control law from (2.8) which requires the integration of a Riccati differential equation over a finite time interval and allows a large class of nonlinearities in the feedback loop without destroying stability. The following definitions are necessary for further analyses.

Definition: The pair $\{A(t),B(t)\}$ is uniformly completely controllable if there exists a positive constant $\delta_c > 0$ such that the following two conditions are satisfied:

(a) $$\alpha_1 I \leq W(t,t+\delta_c) \leq \alpha_2 I, \text{ for all } t \qquad (3.1)$$

(b) $$||\Phi(t,\tau)|| \leq \alpha_3(|t-\tau|), \text{ for all } t,\tau \qquad (3.2)$$

where the controllability matrix $W(t_o,t_1)$ is defined by

$$W(t_o,t_1) = \int_{t_o}^{t_1} \Phi(t_o,t)B(t)B'(t)\Phi'(t_o,t)dt \qquad (3.3)$$

and $\Phi(t,t_o)$ is the state transition matrix for (1.1). Inequalities (3.1) and (3.2) are presumed to hold for some positive constants α_1 and α_2, and for an appropriate function $\alpha_3(\cdot)$ which maps R into R and is bounded on bounded intervals.

Uniform complete observability of the pair $\{A(t),C(t)\}$ is defined similarly with the observability matrix

$$M(t_o,t_1) = \int_{t_o}^{t_1} \Phi'(t,t_o)C'(t)C(t)\Phi(t,t_o)dt \tag{3.4}$$

corresponding to an appropriate $\delta_o > 0$. Let $\delta = \max\{\delta_c,\delta_o\}$. Now consider a matrix Riccati equation

$$-\frac{\partial}{\partial s} P(s,\tau) = -A(s)P(s,\tau) - P(s,\tau)A'(s) - P(s,\tau)C'(s)Q(s)C(s)P(s,\tau)$$

$$+ B(s)R^{-1}(s)B'(s) \tag{3.5}$$

with the boundary condition $P(\tau,\tau) = 0$. The Riccati equation (3.5) comes from the fixed terminal quadratic cost problem (1.1)-(1.4). Note that Equation (2.7) can be obtained from (3.5) by taking $Q(s) = 0$. The following results are necessary for the main theorem.

Lemma 3.1. Assume that $\alpha_4 I \leq R(t) \leq \alpha_5 I$ where α_4 and α_5 are positive constants. (a) Assume that $0 \leq Q(t) \leq \alpha_6 I$ for some $\alpha_6 > 0$. If the pair $\{A(t), B(t)\}$ is uniformly completely controllable and $C(t)$ is bounded, i.e., $||C(t)|| \leq \alpha_7$ for some $\alpha_7 > 0$, then for a fixed T satisfying $\delta_c < T < \infty$ there exist positive constants α_8 and α_9 such that $\tilde{P}(t,t+T)$ obtained from (2.6) and (3.5) satisfies

$$\alpha_8 I \leq \tilde{P}(t,t+T) \leq \alpha_9 I \text{ for all } t. \tag{3.6}$$

(b) Assume that $\alpha_{10} I \leq Q(t) \leq \alpha_6 I$ for some $\alpha_{10} > 0$. If the pairs $\{A(t),B(t)\}$

and $\{A(t),C(t)\}$ are uniformly controllable and observable respectively, then for a fixed T satisfying $\delta < T < \infty$ the relation (3.6) holds.

Proof: (a). It is shown in [2] that there exist positive constants α_{11} and α_{12} such that

$$\alpha_{11} I \leq P(t,t+T) \leq \alpha_{12} I \text{ for all } t. \tag{3.7}$$

The upper bound of (3.6) follows from the fact that

$$\int_t^{t+T} P(t,\tau)d\tau \leq \int_t^{t+T} P(t,t+T)\, d\tau \leq TP(t,t+T) \leq T\,\alpha_{12}$$

since $P(t,\tau_1) \leq P(t,\tau_2)$ for $t \leq \tau_1 \leq \tau_2$. The lower bound follows from

$$\int_t^{t+T} P(t,\tau)\, d\tau = \int_t^{t+\delta_c} P(t,\tau)d\tau + \int_{t+\delta_c}^{t+T} P(t,\tau)d\tau$$

$$\geq \int_t^{t+\delta_c} P(t,\tau)d\tau + \int_{t+\delta_c}^{t+T} P(t,t+\delta_c)d\tau \geq (T - \delta_c)\alpha_{11}.$$

(b) The dual form of inequality (3.7) can be found in [1] and [6]. Inequality (3.6) follows from the above argument. This completes the proof.

By sliding the terminal time of (2.8), i.e. invoking the receding horizon notion, we now introduce a modified control law:

$$u(t) = -R^{-1}(t)B'(t)\tilde{P}^{-1}(t,t+T)x(t),\ T > \delta_c \tag{3.8}$$

where $\tilde{P}(t,t+T) \triangleq \int_t^{t+T} P(t,\tau)d\tau$ and $P(t,\tau)$ is obtained from (2.7) by integrating backward from $s = \tau$ to $s = t$. It is easily seen that (3.8) is the optimal control for the system (1.1) which minimizes the moving cost function

$$\int_t^{t+T}\int_t^{\tau} u'(s)R(s)u(s)dsd\tau \tag{3.9}$$

with the moving boundary condition on the control

$$\int_t^{t+T}\int_t^{\tau} \Phi(t,s)B(s)u(s)ds d\tau = -x(t), \tag{3.10}$$

or equivalently with the moving boundary condition on the state

$$x(t) + \int_t^{t+T} \Phi(t,s)x(s)ds = Tx(t). \tag{3.11}$$

Important properties of the control law (3.8) which will now be demonstrated are (i) the closed-loop system is asymptotically stable, (ii) the gain matrix in (3.8) is obtained from integrating a Riccati equation over a finite time interval, and (iii) the closed-loop system (see Fig. 1) tolerates a large class of nonlinearities in the feedback loop corresponding to a chosen small value of T. In the following theorem the matrix $\tilde{P}(t,t+T)$ in (3.8) is assumed to be obtained from (3.5) instead of (2.7) in order to facilitate a weighting of the state through $Q(t)$.

<u>Theorem 3.1.</u> (a) Assume that $\alpha_4 I \leq R(t) \leq \alpha_5 I$ and $0 \leq Q(t) \leq \alpha_6 I$. If the pair $\{A(t), B(t)\}$ is uniformly completely controllable and $C(t)$ is bounded, then for a fixed T satisfying $\delta_c < T \leq 2$, the system (1.1) with the control law (3.8) is uniformly asymptotically stable where $\tilde{P}(t,t+T)$ is obtained from (3.5). Furthermore, the system (1.1) is uniformly asymptotically stable with either of the following control laws:

(i) $$u(t) = -\psi(t)R^{-1}(t)B'(t)\tilde{P}^{-1}(t,t+T)x(t), T > \delta_c \tag{3.12}$$

where T and the mxm time-varying matrix $\psi(t)$ are such that

$$\psi'(t)R(t) + R(t)\psi(t) \geq TR(t), \tag{3.13}$$

or (ii) $$u(t) = \xi(-R^{-1}(t)B'(t)\tilde{P}^{-1}(t,t+T)x(t)),\ T > \delta_c \tag{3.14}$$

where $\xi(\cdot)$ maps $\mathcal{R}^m$ into $\mathcal{R}^m$ such that

$$w'R(t)\xi(w) \geq \frac{T}{2} w'R(t)w,\ \forall w \in \mathcal{R}^m. \tag{3.15}$$

(b) Assume that $\alpha_4 I \leq R(t) \leq \alpha_5 I$ and $\alpha_{10} I \leq Q(t) \leq \alpha_6 I$. If the pairs $\{A(t), B(t)\}$ and $\{A(t), C(t)\}$ are uniformly completely controllable and observable respectively, then the above results in (a) hold with δ_c replaced by δ.

Proof: Take as a Lyapunov function for the closed-loop system (1.1)-(3.8) with (3.5),

$$V(x,t) = x'\tilde{P}^{-1}(t,t+T)x. \tag{3.16}$$

From Lemma 3.1(a) the function (3.16) has uniform lower and upper bounds. The derivative of (3.16) along the solution is given by

$$\begin{aligned}
\dot{V}(x,t) &= \dot{x}'\tilde{P}^{-1}(t,t+T)x + x'\tilde{P}^{-1}(t,t+T)\dot{x} + x' \frac{d}{dt} \tilde{P}^{-1}(t,t+T)x \\
&= x'\{(A(t)-B(t)\psi(t)R^{-1}(t)B'(t)\tilde{P}^{-1}(t,t+T))'\tilde{P}^{-1}(t,t+T) \\
&\quad + \tilde{P}^{-1}(t,t+T)(A(t)-B(t)\psi(t)R^{-1}(t)B'(t)\tilde{P}^{-1}(t,t+T)) \\
&\quad - \tilde{P}^{-1}(t,t+T)[\frac{d}{dt} \tilde{P}(t,t+T)]\tilde{P}^{-1}(t,t+T)\}\, x.
\end{aligned} \tag{3.17}$$

By integrating both sides of (3.5) we obtain

$$\begin{aligned}
\int_t^{t+T} - \frac{\partial}{\partial t} P(t,\tau)d\tau &= -A(t) \int_t^{t+T} P(t,\tau)d\tau - \int_t^{t+T} P(t,\tau)d\tau\, A'(t) \\
&\quad - \int_t^{t+T} P(t,\tau)C'(t)Q(t)C(t)P(t,\tau)d\tau + \int_t^{t+T} B(t)R^{-1}(t)B'(t)d\tau \\
&= -A(t)\tilde{P}(t,t+T) - \tilde{P}(t,t+T)A'(t) - \int_t^{t+T} P(t,\tau)C'(t)Q(t)C(t)P(t,\tau)d\tau \\
&\quad + T\, B(t)R^{-1}(t)B'(t),
\end{aligned} \tag{3.18}$$

from which follows that

$$\tilde{P}^{-1}(t,t+T)\int_t^{t+T} -\frac{\partial}{\partial t}P(t,\tau)d\tau\ \tilde{P}^{-1}(t,t+T) = -\tilde{P}^{-1}(t,t+T)A(t) - A'(t)\tilde{P}^{-1}(t,t+T)$$

$$+ T\ \tilde{P}^{-1}(t,t+T)B(t)R^{-1}(t)B'(t)\tilde{P}^{-1}(t,t+T)$$

$$-\tilde{P}^{-1}(t,t+T)\int_t^{t+T} P(t,\tau)C'(t)Q(t)C(t)P(t,\tau)d\tau\ \tilde{P}^{-1}(t,t+T). \tag{3.19}$$

Also we have that

$$\frac{d}{dt}\tilde{P}(t,t+T) = \frac{d}{dt}\int_t^{t+T} P(t,\tau)d\tau = P(t,t+T)-P(t,t) + \int_t^{t+T}\frac{\partial}{\partial t}P(t,\tau)d\tau. \tag{3.20}$$

Combining (3.17), (3.19) *and* (3.20) *yields*

$$\dot{V}(x,t) = -\ x'\tilde{P}^{-1}(t,t+T)\ \{P(t,t+T) + \int_t^{t+T}\frac{\partial}{\partial t}P(t,\tau)d\tau - \int_t^{t+T}\frac{\partial}{\partial t}P(t,\tau)d\tau$$

$$+ \int_t^{t+T} P(t,\tau)C'(t)Q(t)C(t)P(t,\tau)d\tau + B(t)R^{-1}(t)[\psi'(t)R(t)$$

$$+ R(t)\psi(t) - TR(t)]R^{-1}(t)B'(t)\}\tilde{P}^{-1}(t,t+T)x$$

$$\leq -\ x'\tilde{P}^{-1}(t,t+T)P(t,t+T)\tilde{P}^{-1}(t,t+T)x$$

provided condition (3.13) holds. From (3.6) and (3.7) it follows that $\dot{V}(x,t) \leq -\alpha_{11}\ \alpha_9^{-2}\ |x|^2$, which implies that the closed-loop system is uniformly asymptotically stable. The proof concerning the control (3.14)-(3.15) is exactly the same as above. With $\psi(t) = I$, or $\xi(w) = w$, it is clear that T must be no greater than 2. The proof of part (b) is also exactly the same as part (a). This completes the proof.

It is noted that nonlinearities satisfying $\psi(t) \geq \frac{T}{2}$ are allowed in the feedback loop for a single input system and that for a given nonlinear function it is possible to find a suitable value of T in order to obtain a stable system. It is also noted that the Lyapunov function (3.16) corresponds to the original system (1.1)-(3.8), while that used for the proof of the control in [2] was defined for the adjoint system in order to include the special case of $Q(t) = 0$. It is interesting to investigate some relationships between the modified control law (3.8) and the open loop control law (2.4) from which the control law (3.8) is obtained. We show in Theorem 3.2 that the double (resp. single) integral quadratic cost for the system (1.1) with the control (3.8) is not larger than the double integral quadratic cost of the open loop control (2.4) with $t_f = t_o + T$ if $t_1 - t_o + T \leq 2$ (resp. $T \leq 1$).

<u>*Theorem 3.2*</u> *The single and double integral quadratic costs for the system (1.1) with the control (3.8) satisfy the following bounds:*

$$\int_{t_o}^{t_1} u'(t)R(t)u(t)\,dt \leq x_o'\tilde{P}^{-1}(t_o,t_o+T)x_o \quad \text{if} \quad T \leq 1, \tag{3.21}$$

$$\int_{t_o}^{t_1}\int_{t_o}^{\tau} u'(t)R(t)u(t)\,dt\,d\tau \leq x_o'\tilde{P}^{-1}(t_o,t_o+T)x_o \quad \text{if} \quad t_1 - t_o + T \leq 2. \tag{3.22}$$

Proof: Let $F(t) = A(t) - B(t)R^{-1}(t)B'(t)\tilde{P}^{-1}(t,t+T)$ and let $\Phi_F(t,t_o)$ be the state transition matrix for $F(t)$. The single integral quadratic cost with the feedback control (3.8) is given by $x_o'N(t_o,t_1)x_o$ where $N(t_o,t_1)$ is the solution of the matrix Riccati equation

$$-\frac{d}{dt}N(t,t_1) = F'(t)N(t,t_1) + N(t,t_1)F(t)$$
$$+ \tilde{P}^{-1}(t,t+T)B(t)R^{-1}(t)B'(t)\tilde{P}^{-1}(t,t+T) \tag{3.23}$$

with the boundary condition $N(t_1,t_1) = 0$. From (3.20) and (3.18) we have

$$- \frac{d}{dt} \tilde{P}^{-1}(t,t+T) = F'(t)\tilde{P}^{-1}(t,t+T) + \tilde{P}^{-1}(t,t+T)F(t) +$$

$$(2-T)\tilde{P}^{-1}(t,t+T)B(t)R^{-1}(t)B'(t)\tilde{P}^{-1}(t,t+T) + \tilde{P}^{-1}(t,t+T)\,[P(t,t+T) +$$

$$\int_t^{t+T} P(t,\tau)C'(t)Q(t)C(t)P(t,\tau)d\tau]\,\tilde{P}^{-1}(t,t+T). \qquad (3.24)$$

Let $E(t) = N(t,t_1) - \tilde{P}^{-1}(t,t+T)$. Then from (3.23) and (3.24) we have

$$- \dot{E}(t) = F'(t)E(t) + E(t)F(t) - (1-T)\tilde{P}^{-1}(t,t+T)B(t)R^{-1}(t)B'(t)\tilde{P}^{-1}(t,t+T)$$

$$- \tilde{P}^{-1}(t,t+T)[P(t,t+T) + \int_t^{t+T} P(t,\tau)C'(t)Q(t)C(t)P(t,\tau)d\tau]\tilde{P}^{-1}(t,t+T)$$

$$(3.25)$$

with the boundary condition $E(t_1) = -\tilde{P}^{-1}(t_1,t_1+T)$. From (3.25) it follows that

$$N(t_o,t_1) - \tilde{P}^{-1}(t_o,t_o+T) = E(t_o) \leq - \Phi_F'(t_1,t_o)\tilde{P}^{-1}(t_1,t_1+T)\Phi_F(t_1,t_o) \qquad (3.26)$$

provided $T \leq 1$. In a similar way it can be shown that the double integral cost with the control (3.8) is given by $x_o'\tilde{N}(t_o,t_1)x_o$ where $\tilde{N}(t_o,t_1)$ is the solution of the matrix Riccati equation

$$- \frac{d}{dt} \tilde{N}(t,t_1) = F'(t)\tilde{N}(t,t_o) + \tilde{N}(t,t_o)F(t)$$

$$+ (t_1 - t)\tilde{P}^{-1}(t,t+T)B(t)R^{-1}(t)B'(t)\tilde{P}^{-1}(t,t+T)$$

with the boundary condition $\tilde{N}(t_1,t_1) = 0$. Let $\tilde{E}(t) = \tilde{N}(t,t_1)-\tilde{P}^{-1}(t,t+T)$.

Then we have

$$-\dot{\tilde{E}}(t) = F'(t)\tilde{E}(t)+\tilde{E}(t)F(t)-(t-t_1 + 2-T)\tilde{P}^{-1}(t,t+T)B(t)R^{-1}(t)B'(t)\tilde{P}^{-1}(t,t+T) - Z(t),$$

where $Z(t)$ is the nonnegative definite last term of (3.25). Thus we have a similar inequality as (3.26) provided $T + t_1 - t_o \leq 2$. This completes the proof.

The relation (3.21) shows that the infinite time control energy is bounded if the system is uniformly completely controllable. The results in this section have simpler forms in the case of time invariant systems as shown in the next section.

IV. Linear Time Invariant Systems

In this section consider a linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_o) = x_o \tag{4.1}$$

$$y(t) = Cx(t) \tag{4.2}$$

where $\{A,B,C\}$ are constant matrices. If the pair $\{A,B\}$ is completely controllable, the minimization of the double integral quadratic control energy (2.1) with a constant matrix R subject to the constraint (2.2) or (2.3) leads to the optimal control law

$$u(t) = -R^{-1}B'\tilde{P}^{-1}(t_f-t)x(t) \tag{4.3}$$

where

$$\tilde{P}(t) = \int_o^t \int_o^\tau e^{-As}BR^{-1}B'e^{-A's}dsd\tau. \tag{4.4}$$

The result analogous to Theorem 3.1 for the time invariant case is contained in the following.

<u>Theorem 4.1</u> If the pair $\{A,B\}$ is completely controllable, then the system (4.1) is uniformly asymptotically stable with the feedback control law

$$u(t) = -R^{-1}B'\tilde{P}^{-1}(T)x(t) \tag{4.5}$$

where $0 \leq T \leq 2$ and $\tilde{P}(T)$ is obtained from

$$\tilde{P}(T) = \int_0^T P(t)dt \tag{4.6}$$

and P(t) is given by

$$\frac{d}{dt} P(t) = -AP(t) - P(t)A' - P(t)C'QCP(t) + BR^{-1}B'$$
$$P(0) = 0 \tag{4.7}$$

for any $Q \geq 0$ and $R > 0$. Furthermore, the system (1.1) is uniformly asymptotically stable with the control law:

$$u(t) = -\psi(t)R^{-1}B'\tilde{P}^{-1}(T)x(t), \quad T > 0 \tag{4.8}$$

where T and $\psi(t)$ satisfy the relation (3.13) with a constant matrix R, or with the control law

$$u(t) = \xi(-R^{-1}B'\tilde{P}^{-1}(T)x(t)), \quad T > 0 \tag{4.9}$$

where T and $\xi(\cdot)$ satisfy the relation (3.15).

From the special structure of a time invariant system, the condition of Theorem 4.1 can be weakened as follows.

Proposition 4.1 If the pair {A,B} is stabilizable, then the system (1.1) is uniformly asymptotically stable with the feedback control

$$u(t) = -R^{-1}B'\tilde{P}^{\dagger}(T)x(t), \quad 0 < T \leq 2 \tag{4.10}$$

where $\tilde{P}^{+}(T)$ is the generalized inverse of the matrix $\tilde{P}(T)$.

The proof of the above result is similar to the one in [2] and is thus omitted here. It is noted that for the special case of $Q = 0$, $\bar{P}(t)$ in (4.3) and (4.5) can be obtained very easily by a single integration as shown below.

Integrating both sides of (4.7) with $Q = 0$ we obtain

$$P(t) = -A\tilde{P}(t) - \tilde{P}(t)A' + tBR^{-1}B'. \tag{4.11}$$

From (4.6) it follows that $P(t) = \frac{d}{dt}\tilde{P}(t)$ so that $\tilde{P}(t)$ satisfies

$$\frac{d}{dt}\tilde{P}(t) = -A\tilde{P}(t) - \tilde{P}(t)A' + tBR^{-1}B' \tag{4.12}$$
$$\tilde{P}(0) = 0.$$

The control (4.5) with $\tilde{P}(T)$ obtained from (4.12) is as easy to obtain as the one in [3], but (4.5) has been shown here to allow a larger class of nonlinearities in the feedback loop.

Some characteristics of the control (4.5) in the frequency domain will now be considered. For notational convenience we define

$$K \triangleq \tilde{P}^{-1}(T), \quad L \triangleq -R^{-1}B'\tilde{P}^{-1}(T),$$

and (4.13)

$$\tilde{Q} \triangleq \tilde{P}^{-1}(T) \int_o^T P(t)C'QCP(t)dt\ \tilde{P}^{-1}(T) + \tilde{P}^{-1}(T)P(T)\tilde{P}^{-1}(T).$$

Then (4.7) is equivalent to the following matrix equation:

$$A'K + KA - T\ KBR^{-1}B'K + \tilde{Q} = 0. \tag{4.14}$$

By a slight modification of the methods introduced in [5], (4.14) can be shown to yield the following relation in the frequency domain:

$$\left[\frac{1}{T} I - R^{\frac{1}{2}} L(-sI - A)^{-1}BR^{-\frac{1}{2}}\right]' \left[\frac{1}{T} I - R^{\frac{1}{2}} L(sI - A)^{-1}BR^{-\frac{1}{2}}\right]$$

$$= \frac{1}{T^2}I + \frac{1}{T} R^{-\frac{1}{2}}B'(-sI - A')^{-1}\ \tilde{Q}(sI - A)^{-1}BR^{-\frac{1}{2}}. \tag{4.15}$$

Relation (4.15) also implies

$$\left[\frac{1}{T} I - R^{\frac{1}{2}}L(-j\omega I - A)^{-1}BR^{-\frac{1}{2}}\right]' \left[\frac{1}{T} I - R^{\frac{1}{2}}L(j\omega I - A)^{-1}BR^{-\frac{1}{2}}\right]$$

$$\geq \frac{1}{T^2} I\ . \tag{4.16}$$

For simplicity we consider a single input system where R can be taken as unity, $B = b$ is a column vector and $L = \ell$ is a row vector. Relation (4.16) can be expressed in this case as follows:

$$\left| \frac{1}{T} - \ell(j\omega I - A)^{-1}b \right| \geq \frac{1}{T} . \qquad (4.17)$$

Inequality (4.17) implies that the Nyquist plot of the closed-loop transfer function $-\ell(j\omega I - A)^{-1}b$ lies outside a circle centered at $-\frac{1}{T} + jo$ with radius $\frac{1}{T}$ as shown in Fig. 2. The key observation, as noted in [5], is that the number of encirclements of any point inside this circle is the same. Theorem 4.1 in conjunction with inequality (4.17) shows that the number of encirclements of points lying on the real axis inside this circle is the same as the number of poles of the transfer function in the closed righthand plane. From these properties and Fig. 2 it is clear that the system employing the control law (4.5) has an infinite gain margin and the following phase margin:

$$\Theta = \tan^{-1} \sqrt{\frac{4}{T^2} - 1} \; . \qquad (4.18)$$

Thus, $\Theta = 0$ for $T = 2$, $\Theta = 60^\circ$ for $T = 1$, and $\Theta \to 90^\circ$ as $T \to 0$. From the well known circle criterion the nonlinear function $\psi(t) \geq \frac{T}{2}$ can be allowed in the closed-loop without destroying its stability. This is consistent with Theorem 4.1.

V. Conclusions

It has been recognized that relatively few optimal control problems lead to stable linear feedback control laws, particularly for time-varying linear systems. In this paper an unconventional optimal control problem has been introduced whose modification leads to a stable control. Since the stabilizing feedback gains are obtained by integrating a Riccati equation over a finite time interval, its computation should be easy with modern computers. The control law (4.5) employing (4.12) is one of the easiest ways to stabilize a linear time invariant system like the one in [3], since the gain matrix can be obtained by a single integration of a Riccati-type equation as indicated in (4.12). The results indicate that a short receding horizon distance T should be chosen when large nonlinearities exist in the feedback loop. Stabilization with a prescribed degree of stability can be obtained exactly the same as in [2].

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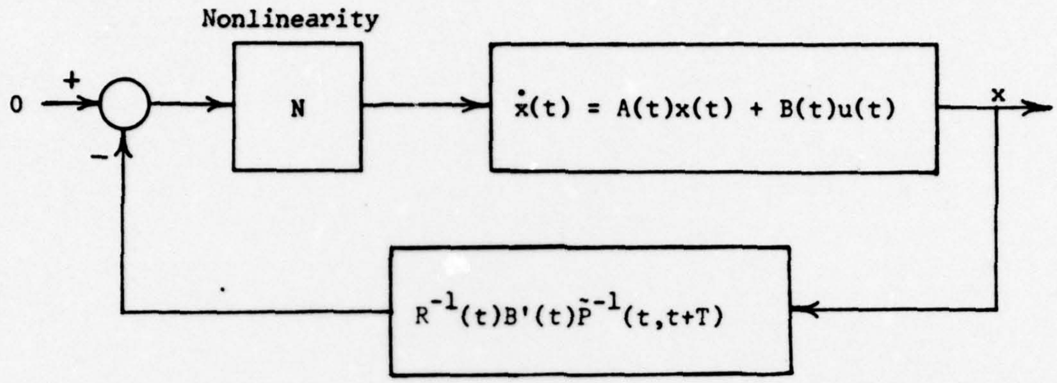

Fig. 1. Nonlinearity in the Feedback Loop

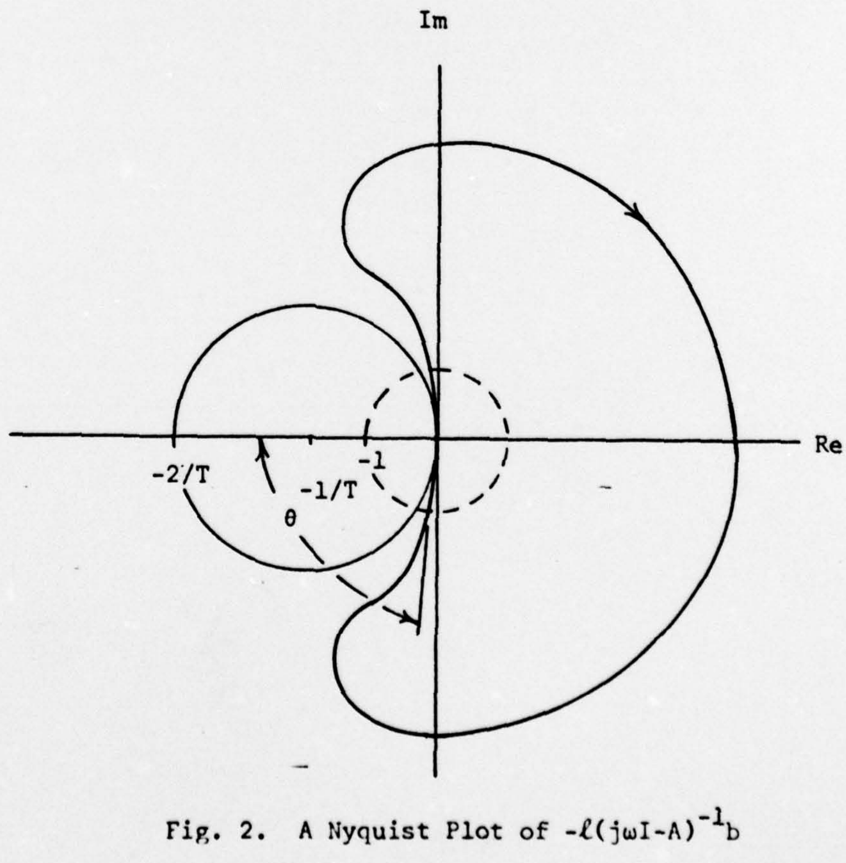

Fig. 2. A Nyquist Plot of $-\ell(j\omega I-A)^{-1}b$

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This paper considers an optimal regulator problem which is different from the conventional linear quadratic cost problem but leads to a stable linear feedback control. The problem considers a double integral quadratic cost function and an integral constraint on state trajectories. The optimal

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20. ABSTRACT (Continued)

open-loop control is transformed to a closed-loop control and subsequently a modified control is obtained based on a receding horizon notion. This modified control law is shown to be asymptotically stable and to result in a new method for stabilizing linear time-varying systems, in addition to the methods of [1] and [2], as well as providing an easy means to stabilize time-invarient systems, comparable to the method of [3]. Moreover, the gain matrix for the modified control is obtained from a Riccati-type equation over a finite time interval, and a large class of nonlinearities can be allowed in the closed-loop without destroying its stability.